8. INDUCING CHARACTERS

§ 8.1. Recycling Characters

In this section we learn some techniques for constructing new characters out of existing ones.

Theorem 1: The sum of two characters is a character.

Proof: If ρ and σ are representations for G then $(\rho+\sigma)(g) = \begin{pmatrix}\rho(g) & 0 \\ 0 & \sigma(g)\end{pmatrix}$ is a representation of G and the trace of $\begin{pmatrix}\rho(g) & 0 \\ 0 & \sigma(g)\end{pmatrix}$ is the sum of the traces of $\rho(g)$ and $\sigma(g)$. ☺✋

Theorem 2:
The conjugate of a character is a character.
The conjugate of an irreducible character is irreducible.

Proof: If ρ is a representation over ℂ then
$\rho^*(g) = \rho(g^{-1})^T$ is a representation because
$\rho^*(gh) = \rho(g^{-1})^T\rho(h^{-1})^T = [\rho(h^{-1})\rho(g^{-1})]^T$
$= \rho(h^{-1}g^{-1})^T = \rho((gh)^{-1})^T = \rho^*(gh)$.

There are three stages in the definition of ρ*: ρ itself, inverting and transposing. All three take products to products but two of them, inverting and transposing, change the order of the factors. Of course swapping the factors twice ensures that everything comes out in the right order.

The second part follows from the fact that a character χ is irreducible if and only if $\langle\chi|\chi\rangle = 1$. ☺🖑

Theorem 3:
The product of two characters is a character.
The product of an irreducible character by a linear character is irreducible.
Proof: The first part is true in all cases but we'll prove it only in the case where one of the characters is linear. Suppose ρ, σ are representations where σ is linear. Being linear, the character of σ is σ itself. Let χ be the character of ρ.

Now $(\sigma\rho)(g) = \sigma(g)\rho(g)$ is a representation because

$$\sigma(gh)\rho(gh) = \sigma(g)\sigma(h)\rho(g)\rho(h)$$
$$= \sigma(g)\rho(g)\sigma(h)\rho(h).$$

The commuting of $\sigma(h)$ with $\rho(g)$ is because $\sigma(h)$ is a scalar. Again, since σ is linear, the trace of $\sigma(g)\rho(g)$ is $\sigma(g)\chi(g)$.

For the second part note that for every $g \in G$, $\sigma(g)$ is a root of unity and so $|\sigma(g)| = 1$.

Thus $\langle\sigma\chi|\sigma\chi\rangle = \frac{1}{|G|}\sum_{g\in G}|\sigma(g)|^2|\chi(g)|^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 1$.

☺🖑

Example 1: In Example 13, $\chi_2 + \chi_3$ is the character

$$[2, 2, -1, -1].$$

Being the sum of two characters it isn't irreducible which is why it doesn't appear in the character table. The product of χ_2 and χ_3 is χ_1, and χ_3 is the conjugate of χ_2.

Multiplying χ_4 by the three irreducible linear characters χ_1, χ_2, χ_3 would appear to give three irreducible characters of degree 3. In fact it does, but because of the location of the 0's they're all equal – not very useful. But if we hadn't yet found χ_4 we could infer, that since there's only one irreducible character of degree 3 the entries in χ_4 for columns where χ_1, χ_2, χ_3 differ, must be zero. (Of course we could have completed the last row by orthogonality.)

§ 8.2. Cyclic Groups

If $\theta = e^{2\pi i/n}$, and G is the cyclic group $\langle A \mid A^n \rangle$, the map $A^k \to \theta^k$ is an isomorphism and hence a faithful representation. Being a linear representation it is its own character. So we get a character χ such that $\chi(A^k) = \theta^k$. Powers of χ give all the other irreducible characters.

In the previous chapter we calculated the character tables of $\mathbf{C}_2$ and $\mathbf{C}_3$. The character table of $\mathbf{C}_n = \langle A \mid A^n \rangle$ is $n \times n$. If Γ_n is the conjugacy class $\{A^n\}$ then $\chi_i(\Gamma_j) = \theta^{ij}$ where $\theta = e^{2\pi i/n}$. Here is the character table for $\mathbf{C}_4$.

Example 4: $C_4 = \langle A | A^4 \rangle$

class	1	A	A^2	A^3
size	1	1	1	1
χ_1	1	1	1	1
χ_2	1	i	–1	–i
χ_3	1	–1	1	–1
χ_4	1	–i	–1	i
order	1	4	2	4

§ 8.3. Inducing Up from Quotient Groups

One of the most useful techniques for completing character tables is to find a character of a quotient group and to lift it up to the whole group.

Theorem 4: Every irreducible character of G/H induces an irreducible character of G.

Proof: Suppose ρ is a representation for G/H. This induces a representation Ψ for G by defining

$$\Psi(g) = \rho(gH).$$

This is a representation because

$$\begin{aligned} \Psi(ab) &= \rho(ab\mathrm{H}) \\ &= \rho(a\mathrm{H}.b\mathrm{H}) \\ &= \rho(a\mathrm{H}).\rho(bH) \\ &= \Psi(a)\Psi(b). \end{aligned}$$

If G has many normal subgroups a large part of the character table for G can be induced from those of smaller quotient groups.

Example 5: The group in Example 13 is G = $\mathbf{A}_4$, the group of all even permutations on the set {1, 2, 3, 4}. Here's how we can go about constructing its character table.

Suppose we've found that there are 4 conjugacy classes, {I}, one class consisting of the three permutations with cycle structure (××)(××) and two classes of size 4 containing between them the 8 cycles of length 3.

Now G has a normal subgroup H of order 4, consisting of the first two conjugacy classes and G/H is of order 3. So G/H must be a cyclic group of order 3. Let C be a generator. G/H has character table:

class	**1**	**C**	$\mathbf{C^2}$
size	**1**	**1**	**1**
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω
order	**1**	**3**	**3**

Now we induce up to G. Classes Γ_1, Γ_2 for G correspond to 1 in G/H. Class Γ_3 in G corresponds to C in G/H and Γ_4 in G corresponds to C^2. So inducing up from the three irreducible characters of G/H to G we get three of the four irreducible characters of G:

in G/H	1		C	C^2
class	Γ_1	Γ_2	Γ_3	Γ_4
size	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4				
order	1	2	3	3

The degree of χ_4 is 3 ($1^2 + 1^2 + 1^2 + 3^2 = 12$). We can find the remaining values of χ_4 by orthogonality of the columns. So χ_4 is (3, –1, 0, 0).

§ 8.4. Direct Products

Let $G = H \times K$ be the direct product of H, K.
Let the conjugacy classes of H and K be $\Gamma_1, \Gamma_2, \ldots, \Gamma_h$ and $\Omega_1, \Omega_2, \ldots, \Omega_k$ respectively. Then the conjugacy classes of G are $\{\Gamma_i \times \Omega_j\}$.

If the character tables for H, K are $\chi = (\chi_{ij})$ and $\Omega = (\Omega_{ij})$ then the character table for $G \times H$ is:

$$\begin{pmatrix} \chi_{11}\Omega & \ldots & \chi_{1n}\Omega \\ \ldots & \ldots & \ldots \\ \chi_{n1}\Omega & \ldots & \chi_{nn}\Omega \end{pmatrix}$$

Example 6: The character table for $\mathbf{A}_4$ is

class	**I**	**(××)(××)**	**(×××)**	**(×××)**
size	**1**	**3**	**4**	**4**
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	–1	0	0
order	**1**	**2**	**3**	**3**

and for $\mathbf{S}_3$ it is

class	**I**	**(×××)**	**(××)**
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
order	**1**	**3**	**2**

If M is the character table for $\mathbf{A}_4$ then the character table for $\mathbf{A}_4 \times \mathbf{S}_3$ is:

	$\mathbf{A}_4 \times$ **I**	$\mathbf{A}_4 \times$ **(×××)**	$\mathbf{A}_4 \times$ **(××)**
χ_1 - χ_4	M	M	M
χ_5 - χ_8	M	M	–M
χ_9 - χ_{12}	2M	–M	0

When written out in full it's rather large so I've had to split it horizontally.

To save space I’ve used abbreviated notation for cycle structures. So, for example, 3 × 22 represents the conjugacy class (×××) × (××)(××), where (×××) comes from $\mathbf{S}_3$ and (××)(××) comes from $\mathbf{A}_4$. Writing it out in full, and using ‘[‘ and ‘]’ for the ordered pairs to avoid confusion, 3×22 represents the conjugacy class:

{[(12), (12)(34)], [(12), (13)(24)], [(12), (13)(24)],
{[(13), (12)(34)], [(13), (13)(24)], [(13), (13)(24)],
{[(23), (12)(34)], [(23), (13)(24)], [(23), (13)(24)]}

of size 6. Notice that there are two conjugacy classes labelled 3 × 3 because there are two conjugacy classes in $\mathbf{A}_4$ that contain the 3-cycles.

class	**I×I**	**I×22**	**I×3**	**I×3**	**3×I**	**3×22**	**3×3**	**3×3**
size	**1**	**3**	**4**	**4**	**2**	**6**	**8**	**8**
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	ω	ω^2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω	1	1	ω^2	ω
χ_4	3	–1	0	0	3	–1	0	0
χ_5	1	1	1	1	1	1	1	1
χ_6	1	1	ω	ω^2	1	1	ω	ω^2
χ_7	1	1	ω^2	ω	1	1	ω^2	ω
χ_8	3	–1	0	0	3	–1	0	0
χ_9	2	2	2	2	–1	–1	–1	–1
χ_{10}	2	2	2ω	$2\omega^2$	–1	–1	$-\omega$	$-\omega^2$
χ_{11}	2	2	$2\omega^2$	2ω	–1	–1	$-\omega^2$	$-\omega$
χ_{12}	6	–2	0	0	–3	1	0	0
order	**1**	**2**	**3**	**3**	**3**	**6**	**3**	**3**

Now for the last 4 columns in the table.

class	2×I	2×22	2×3	2×3
size	3	9	12	12
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	–1	0	0
χ_5	–1	–1	–1	–1
χ_6	–1	–1	$-\omega$	$-\omega^2$
χ_7	–1	–1	$-\omega^2$	$-\omega$
χ_8	–3	1	0	0
χ_9	0	0	0	0
χ_{10}	0	0	0	0
χ_{11}	0	0	0	0
χ_{12}	0	0	0	0
order	2	2	6	6

§ 8.5. Abelian Groups

In a later chapter I'll show that finite abelian groups are direct sums of cyclic groups and so the above technique can be used to construct their character tables.

Example 7: Find the character table of $\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_3$.
Solution: The character table of $\mathbf{C}_2 \times \mathbf{C}_2$ is

χ_1	1	1	1	1
χ_2	1	–1	1	–1
χ_3	1	1	–1	–1
χ_4	1	–1	–1	1
order	1	2	2	2

and the character table for $\mathbf{C}_3$ is:

χ_1	1	1	1
χ_2	1	ω	ω^2
χ_4	1	ω^2	ω
order	**1**	**3**	**3**

The character table of $\mathbf{C}_2 \times \mathbf{C}_2$ is therefore:

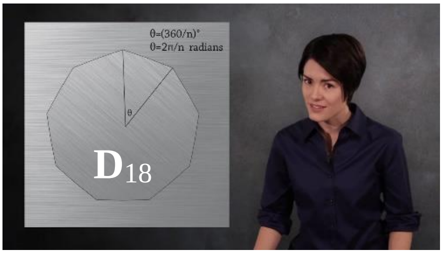
χ_1	1	1	1	1
χ_2	1	–1	1	–1
χ_3	1	1	–1	–1
χ_4	1	–1	–1	1
order	**1**	**2**	**2**	**2**

Hence the character table for $\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_3$ is as follows. (Again, I've had to split the table vertically.)

χ_1	1	1	1	1	1	1	1	1
χ_2	1	–1	1	–1	1	–1	1	–1
χ_3	1	1	–1	–1	1	1	–1	–1
χ_4	1	–1	–1	1	1	–1	–1	1
χ_5	1	1	1	1	ω	ω	ω	ω
χ_6	1	–1	1	–1	ω	$-\omega$	ω	$-\omega$
χ_7	1	1	–1	–1	ω	ω	$-\omega$	$-\omega$
χ_8	1	–1	–1	1	ω	$-\omega$	$-\omega$	ω
χ_9	1	1	1	1	1	ω^2	ω^2	ω^2
χ_{10}	1	–1	1	–1	1	$-\omega^2$	ω^2	$-\omega^2$
χ_{11}	1	1	–1	–1	1	ω^2	$-\omega^2$	$-\omega^2$
χ_{12}	1	–1	–1	1	1	$-\omega^2$	$-\omega^2$	ω^2
order	**1**	**2**	**2**	**2**	**3**	**6**	**6**	**6**

χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1
χ_5	ω^2	ω^2	ω^2	ω^2
χ_6	ω^2	$-\omega^2$	ω^2	$-\omega^2$
χ_7	ω^2	ω^2	$-\omega^2$	$-\omega^2$
χ_8	ω^2	$-\omega^2$	$-\omega^2$	ω^2
χ_9	ω^2	ω	ω	ω
χ_{10}	ω^2	$-\omega$	ω	$-\omega$
χ_{11}	ω^2	ω	$-\omega$	$-\omega$
χ_{12}	ω^2	$-\omega$	$-\omega$	ω
order	**3**	**6**	**6**	**6**

§ 8.6. Dihedral Groups



In the previous chapter we found the character table for $\mathbf{C}_2$, $\mathbf{V}_4$ and $\mathbf{S}_3$. These are the first few dihedral groups, with $\mathbf{D}_2 \cong \mathbf{C}_2$, $\mathbf{D}_4 \cong \mathbf{V}_4 \cong \mathbf{C}_2 \times \mathbf{C}_2$ and $\mathbf{D}_6 \cong \mathbf{S}_3$.

Here is $\mathbf{D}_8$.

Example 8: $\mathbf{D_8} = \langle A, B \mid A^4, B^2, B^{-1}AB = A^{-1}\rangle$
The conjugacy classes are: $\{1\}$, $\{A^2\}$, $\{A, A^3\}$, $\{B, A^2B\}$, $\{AB, A^3B\}$. $H = \langle A^2\rangle$ is a normal subgroup of order 2 and $G/H \cong \mathbf{C}_2 \oplus \mathbf{C}_2$.

class	1	A	A^2	B	AB
size	1	2	1	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	–1	–1
χ_3	1	–1	1	1	–1
χ_4	1	–1	1	–1	1
χ_5	2	0	–2	0	0
order	1	4	2	2	2

NOTES:

χ_1 is the trivial character.

χ_2, χ_3 and χ_4 can be induced from $G/\langle A^2\rangle$.

χ_5 can be obtained by orthogonality

The general dihedral group is:

$$\mathbf{D}_{2n} = \langle A, B \mid A^n, B^2, B^{-1}AB = A^{-1}\rangle.$$

If n is odd the class equation is:

$$2n = 1 + 2 + 2 + .. + 2 + n$$

and there are ½(n + 3) irreducible characters:

2 linear and ½(n – 1) of degree 2.

If n is even the class equation is:

$$2n = 1 + 1 + 2 + 2 + .. + 2 + \tfrac{1}{2}n + \tfrac{1}{2}n$$

and there are ½(n + 6) irreducible characters:

4 linear and ½(n – 2) of degree 2.

§ 8.7. Symmetric Groups

In the previous chapter we found the character table for $\mathbf{S}_3$. Here is $\mathbf{S}_4$. Unlike the larger symmetric groups $\mathbf{S}_4$ has some convenient normal subgroups.

Example 9: $\mathbf{S_4}$ The trivial character and the permutation character are:

class	I	(××)(××)	(×××)	(××)	(××××)
size	1	3	8	6	6
χ_1	1	1	1	1	1
Π	4	0	1	2	0

Since $\langle\Pi|\Pi\rangle = 2 = 1^2 + 1^2$, Π must be the sum of two irreducible characters.
Since $\langle\Pi|\chi_1\rangle = 1$, Π must be χ_1 plus another irreducible character. So $\Pi - \chi_1$ is an irreducible character. Thus we can complete the character table for $\mathbf{S}_4$.

class	I	(××)(××)	(×××)	(××)	(××××)
size	1	3	8	6	6
χ_1	1	1	1	1	1
χ_2	1	1	1	–1	–1
χ_3	2	2	–1	0	0
χ_4	3	–1	0	–1	1
χ_5	3	–1	0	1	–1
order	1	2	3	2	4

NOTES:

χ_1 is the trivial character.

χ_2 can be induced from G/$\mathbf{A}_4$.

χ_3 can be obtained by orthogonality.

$\chi_4 = \chi_2\chi$.

$\chi_5 = \Pi - \chi_1$

§ 8.8. Inducing Up from Subgroups

Inducing up from a quotient group is of no use if we want to find the character table of a simple group – one with no proper, non-trivial normal subgroup. But simple groups have plenty of subgroups, and it's possible to induce up from them. The big difference between the subgroup and quotient group situation is that when we induce up from an irreducible character of a subgroup we rarely get an irreducible character. Extra work is needed to split them into irreducible characters.

Theorem 5: If H ≤ G and χ is a character of H and g lies in the conjugacy class Γ (of G) then the value of the induced character on g is $\frac{|G|}{|H|} \times \frac{|\Gamma \cap H|}{|\Gamma|} \times \frac{\sum_{h \in \Gamma \cap H} h\chi}{|\Gamma \cap H|}$.

Proof: We omit the proof of this theorem. ☺

We can remember this theorem by saying that:

Inducing up from H on the class Γ:
the index of H × the proportion of Γ in H × the average character for these.

or more simply:

index × proportion × average

Example 10: In addition to $\mathbf{S}_3$ being isomorphic to a quotient group of $\mathbf{S}_4$ it is also a subgroup. In fact there are four subgroups of S_4 that are isomorphic to $\mathbf{S}_3$:

$$H_n = \{g \in \mathbf{S}_4 \mid g(n) = n\} \text{ for } n = 1, 2, 3, 4.$$

Let's take H_4 as our $\mathbf{S}_3$ so that we can use the usual notation for $\mathbf{S}_3$.

The conjugacy classes of $\mathbf{S}_3$ are $\Omega_1 = \{I\}$, $\Omega_2 = \{(12), (13), (23)\}$ and $\Omega_3 = \{(123), (132)\}$.

The character table, over $\mathbb{C}$, of $\mathbf{S}_3$ is:

class	$\mathbf{\Omega_1}$	$\mathbf{\Omega_2}$	$\mathbf{\Omega_3}$
size	**1**	**3**	**2**
χ_1	1	1	1
χ_2	1	–1	1
χ_3	2	0	–1

The conjugacy classes of $\mathbf{S}_4$ are
$\Gamma_1 = \{I\}$,
$\Gamma_2 = (\times\times)$
$= \{(12), (13), (14), (23), (24), (34)\}$,
$\Gamma_3 = (\times\times\times)$
$= \{(123), (132), (124), (142), (134), (143), (234), (243)\}$,

$\Gamma_4 = (\times\times\times\times)$
$\quad = \{(1234), (1243), (1324), (1242), (1423), (1432)\}$ and
$\Gamma_5 = (\times\times)(\times\times)$
$\quad = \{(12)(34), (13)(24), (14)(23)\}$.

The index of $\mathbf{S}_3$ in $\mathbf{S}_4$ is 24/6 = 4.
The proportions of $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ that lie in $\mathbf{S}_3$ are:
1, ½, ¼, 0, 0 respectively.
We now induce the irreducible characters χ_1, χ_2, χ_3 of $\mathbf{S}_3$ to give the characters $\theta_1, \theta_2, \theta_3$ of $\mathbf{S}_4$.
For example, θ_3 on Ω_3 gives the value $4 \times ¼ \times -1$.
These three characters of $\mathbf{S}_4$ are as follows.

class	$\mathbf{\Omega_1}$	$\mathbf{\Omega_2}$	$\mathbf{\Omega_3}$	$\mathbf{\Omega_4}$	$\mathbf{\Omega_5}$
size	**1**	**6**	**8**	**6**	**3**
$\boldsymbol{\theta_1}$	4	2	1	0	0
$\boldsymbol{\theta_2}$	4	–2	1	0	0
$\boldsymbol{\theta_3}$	8	0	–1	0	0

None of these induced characters are irreducible. We can see this by computing inner products.

$$\langle\theta_1|\theta_1\rangle = \frac{4^2.1 + 2^2.6 + 1^2.8}{24} = \frac{16 + 24 + 8}{24} = 2.$$

This is the sum of squares of the multiplicities of irreducible characters when θ_1 is expressed as a sum of irreducibles. Hence θ_1 is the sum of two irreducible characters.

The inner product of θ_1 and the trivial character of **S**$_4$ is $\frac{4.1 + 2.6 + 1.8}{24} = 1$, so the trivial character of **S**$_4$ is one of the two irreducibles referred to above. This entitles us to subtract the trivial character from θ_1, giving another irreducible character for **S**$_4$.

class	$\mathbf{\Omega_1}$	$\mathbf{\Omega_2}$	$\mathbf{\Omega_3}$	$\mathbf{\Omega_4}$	$\mathbf{\Omega_5}$
size	1	6	8	6	3
trivial	1	1	1	1	1
	3	1	0	–1	–1

$\langle\theta_3|\theta_3\rangle = \frac{8^2.1 + 1^2.8}{24} = \frac{64 + 8}{24} = 3$. Hence θ_3 is the sum of three irreducible characters.

The inner product of θ_3 and the trivial character of S$_4$ is $\frac{8.1 + (-1).8}{24} = 0$, so the trivial character of **S**$_4$ is *not* one of these three irreducible characters. We are therefore *not* entitled to subtract the trivial character from θ_1.

The inner product of θ_3 and the irreducible character we obtained earlier is $\frac{8.3.1}{24} = 1$, so we are permitted to subtract, giving:

class	$\mathbf{\Omega_1}$	$\mathbf{\Omega_2}$	$\mathbf{\Omega_3}$	$\mathbf{\Omega_4}$	$\mathbf{\Omega_5}$
size	**1**	**6**	**8**	**6**	**3**
	5	–1	–1	1	1

This is not irreducible, but is the sum of two irreducible characters, neither of which are the two we already have. There are three irreducible characters yet to be found, and we know the sum of two of them. However this is as far as we can go inducing up from the subgroup $\mathbf{S}_3$.

Picking another subgroup, such as the cyclic group generated by (1234), we could get additional information. But we've already found the character table of $\mathbf{S}_4$ using the simpler technique of inducing up from quotient groups.

Inducing up from a quotient group one should know the entire character table of the quotient. Inducing up from the trivial character of the quotient only would be a waste of time because it will only give you the trivial character of the whole group.

But inducing up from the trivial character of a subgroup can be quite useful. It's usually not irreducible, and so one may have to subtract off characters to obtain a useful irreducible. Often, inducing up from the trivial characters of various subgroups will give enough information to complete the character table of a group.

Example 12: Find the character table of $\mathbf{S}_5$.

Solution: The conjugacy classes correspond to the cycle structures: I, (××), (×××), (××××), (×××××), (××)(××) and (×××)(××). We can abbreviate these by just writing down the sizes of the cycles. Of course the identity has to be special and we simply denote {I} by I itself. So I'll denote the conjugacy classes as follows: I, 2, 3, 4, 5, 2.2 and 3.2 respectively.

The only useful normal subgroup of $\mathbf{S}_5$ is $\mathbf{A}_5$ and inducing up from $\mathbf{S}_5/\mathbf{A}_5$ only gives 2 of the 7 irreducible characters of $\mathbf{S}_5$.

	I	**2**	**3**	**4**	**5**	**2.2**	**3.2**
	1	**10**	**20**	**30**	**24**	**15**	**20**
χ_1	1	1	1	1	1	1	1
χ_2	1	–1	1	–1	1	1	–1

We need to induce up from subgroups. And we need to use large subgroups so that the index is small. $\mathbf{A}_5$ might sound useful in that it gives a degree 2 character. But it will only give you the sum of the two linear characters that we already have.

A subgroup would be $\mathbf{S}_4$. By this I mean the subgroup that permutes 1, 2, 3 and 4 and fixes 5. Another large subgroup that has small index is $\mathbf{S}_3 \times \mathbf{S}_2$, by which I mean the subgroup consisting of permutations that permute 1, 2 and 3 and, independently, 4 and 5. An example of a permutation in the $\mathbf{S}_3 \times \mathbf{S}_2$ subgroup would be (132)(45).

Inducing up from the trivial character of $\mathbf{S}_4$ the index is 5 and the average value of the trivial character is 1. (In most cases the average is simply *the* value. The only time we have to do an actual average is where a conjugacy class in G splits into two or more inside the subgroup and where the values of the character differ. Clearly this will never happen if we are inducing up from the trivial character.) So each value will be 5 times the proportion of the conjugacy class that lies in S_4.

I: The proportion is clearly 1.

(××): There are 6 of these in $\mathbf{S}_4$ so the proportion is $\frac{6}{10}$.

(×××): There are 8 of these in $\mathbf{S}_4$ so the proportion is $\frac{8}{20}$.

(××××): There are 6 of these in $\mathbf{S}_4$ so the proportion is $\frac{6}{30}$.

(×××××): None of these are in $\mathbf{S}_4$ so the proportion is 0.

(××)(××): There are 3 of these in $\mathbf{S}_4$ and so the proportion is $\frac{3}{15}$.

(×××)(××): None of these are in $\mathbf{S}_4$, so the proportion is 0.

Multiplying each of these by the index 5 we get the character:

	I	**2**	**3**	**4**	**5**	**2.2**	**3.2**
	1	**10**	**20**	**30**	**24**	**15**	**20**
$\boldsymbol{\theta_1}$	5	3	2	1	0	1	0

$\langle\theta_1|\theta_1\rangle = 2$ so θ_1 must be the sum of two irreducible characters. $\langle\theta_1|\chi_1\rangle = 1$ so $\theta_1 = \chi_1$ plus another irreducible character. Subtracting χ_1 this gives us an irreducible character of degree 4.

	I	**2**	**3**	**4**	**5**	**2.2**	**3.2**
	1	**10**	**20**	**30**	**24**	**15**	**20**
χ_1	1	1	1	1	1	1	1
χ_2	1	–1	1	–1	1	1	–1
χ_3	4	2	1	0	–1	0	–1

Now we induce up from the trivial character of $\mathbf{S}_3 \times \mathbf{S}_2$. The index is 10. Again we only need to work out the proportions.

I: The proportion is clearly 1.

(××): There are 4 of these and so the proportion is $\frac{4}{10}$.

(×××): There are 2 of these and so the proportion is $\frac{2}{20}$.

(××××): There are 0 of these and so the proportion is 0.

(×××××): There are 0 of these and so the proportion is 0.

(××)(××): There are 3 of these and so the proportion is $\frac{3}{15}$.

(×××)(××): There are 2 of these, so the proportion is $\frac{2}{20}$.

Multiplying each of these by 10 we get the character:

	I	**2**	**3**	**4**	**5**	**2.2**	**3.2**
	1	**10**	**20**	**30**	**24**	**15**	**20**
θ_2	10	4	1	0	0	2	1

$\langle\theta_2|\theta_2\rangle = 3$ so θ_2 is the sum of two irreducible characters.

$\langle\theta_2|\chi_1\rangle = 1$ so $\theta_1 = \chi_1$ plus the sum of two other irreducible character.
$\langle\theta_2|\chi_3\rangle = 1$ so $\theta_1 = \chi_3$ plus the sum of two other irreducible character.
Hence $\chi_4 = \theta_2 - \chi_1 - \chi_3$ is irreducible character of degree 5.

And of course $\chi_3\chi_2$ and $\chi_4\chi_2$ will give two further irreducible characters χ_5 and χ_6. There is only one irreducible character to go, and this can be found by orthogonality.

Hence the character table for $\mathbf{S}_5$ is

	I	**2**	**3**	**4**	**5**	**2.2**	**3.2**
size	**1**	**10**	**20**	**30**	**24**	**15**	**20**
χ_1	1	1	1	1	1	1	1
χ_2	1	–1	1	–1	1	1	–1
χ_3	4	2	1	0	–1	0	–1
χ_4	5	1	–1	–1	0	1	1
χ_5	4	–2	1	0	–1	0	1
χ_6	5	–1	–1	1	0	1	–1
χ_7	6	0	0	0	1	–2	0
order	**1**	**2**	**3**	**4**	**5**	**2**	**6**

EXERCISES FOR CHAPTER 8

Exercise 1:

The following is a character of $\mathbf{C}_3 = \langle A | A^3 = 1 \rangle$.

class	1	A	A^2
size	1	1	1
θ	4	$2\omega^2 - 1$	$2\omega - 1$
order	1	3	3

(a) Find the conjugate of θ;

(b) Calculate $\langle \theta | \theta \rangle$;

(c) Express θ as a sum of irreducible characters, using the notation of Example 3.

Exercise 2:

(a) Construct the character table for $\mathbf{C}_6$ in two ways:

(i) as $\langle A \mid A^6 = 1 \rangle$ and

(ii) as $\langle A, B \mid B^3 = C^2 = 1, BC = CB \rangle$.

(b) Reconcile these two different-looking answers.

Exercise 3:

Construct the character table for $\mathbf{S}_3 \times \mathbf{D}_8$.

Exercise 4: Let $\chi_1, \ldots, \chi_5$ be the irreducible characters of $\mathbf{D}_8$ in Example 8.

Calculate the characters: $\chi_2 + \chi_3 + \chi_4$ and $2\chi_4 + \chi_5$.

Exercise 5: Examine the character table for $\mathbf{S}_4$ in Example 9. Find χ_5^2 and express it as a sum of irreducible characters.

Exercise 6: The dihedral group

$$\mathbf{D}_8 = \langle A, B | A^4 = B^2 = 1, BA = A^{-1}B\rangle$$

has the subgroup $H = \langle B \rangle$.

(a) Induce both irreducible characters of H up to characters θ_1, θ_2 for $\mathbf{D}_8$.

(b) Write each of θ_1 and θ_2 as a sum of irreducible characters. (Use the notation of Example 8.)

SOLUTIONS FOR CHAPTER 8

Exercise 1:

(a)

class	1	A	A^2
size	1	1	1
$\bar{\theta}$	4	$2\omega - 1$	$2\omega^2 - 1$
order	1	3	3

(b) $$\langle \theta | \theta \rangle = \frac{16 + (2\omega - 1)(2\omega^2 - 1) + (2\omega^2 - 1)(2\omega - 1)}{3}$$

$$= \frac{16 + 2(4 - 2\omega - 2\omega^2 + 1)}{3}$$

$$= \frac{16 + 2(4 + 2 + 1)}{3}$$

$$= \frac{30}{3} = 10.$$

(c) The only possibility of expressing 10 as a sum of 3 squares (there are 3 irreducible characters) is $3^2 + 1^2 + 0^2$. Hence θ is 3 times one irreducible character plus a second. It is easily checked that $\theta = \chi_2 + 3\chi_3$.

Exercise 2:

Let $\theta = e^{2\pi/6}$. Then the character table is:

class	**1**	**A**	$\mathbf{A^2}$	$\mathbf{A^3}$	$\mathbf{A^4}$	$\mathbf{A^5}$
χ_1	1	1	1	1	1	1
χ_2	1	θ	θ^2	θ^3	θ^4	θ^5
χ_3	1	θ^2	θ^4	θ^6	θ^8	θ^{10}
χ_4	1	θ^3	θ^6	θ^9	θ^{12}	θ^{15}
χ_5	1	θ^4	θ^8	θ^{12}	θ^{16}	θ^{20}
χ_6	1	θ^5	θ^{10}	θ^{15}	θ^{20}	θ^{25}
order	**1**	**6**	**3**	**2**	**3**	**6**

Simplifying we get:

class	**1**	**A**	$\mathbf{A^2}$	$\mathbf{A^3}$	$\mathbf{A^4}$	$\mathbf{A^5}$
χ_1	1	1	1	1	1	1
χ_2	1	θ	θ^2	–1	θ^4	θ^5
χ_3	1	θ^2	θ^4	1	θ^2	θ^4
χ_4	1	–1	1	–1	1	–1
χ_5	1	θ^4	θ^2	1	θ^4	θ^2
χ_6	1	θ^5	θ^4	–1	θ^2	θ
order	**1**	**6**	**3**	**2**	**3**	**6**

The character table for $\mathbf{C}_3 \times \mathbf{C}_2$ is:

class	1	B	B^2	C	BC	B^2C
χ_1	1	1	1	1	1	1
χ_2	1	ω	ω^2	1	ω	ω^2
χ_3	1	ω^2	ω	1	ω^2	ω
χ_4	1	1	1	–1	–1	–1
χ_5	1	ω	ω^2	–1	$-\omega$	$-\omega^2$
χ_6	1	ω^2	ω	–1	$-\omega^2$	$-\omega$
order	**1**	**3**	**3**	**2**	**6**	**6**

(c) Writing ω as θ^2, $-\omega$ as θ^5, ω^2 as θ^4 and $-\omega^2$ as θ the above table becomes:

class	1	B	B^2	C	BC	B^2C
χ_1	1	1	1	1	1	1
χ_2	1	θ^2	θ^4	1	θ^2	θ^4
χ_3	1	θ^4	θ^2	1	θ^4	θ^2
χ_4	1	1	1	–1	–1	–1
χ_5	1	θ^2	θ^4	–1	θ^5	θ
χ_6	1	θ^4	θ^2	–1	θ	θ^5
order	**1**	**3**	**3**	**2**	**6**	**6**

Rearranging rows and columns this becomes the same table as for $\langle A | A^6 = 1 \rangle$.

Exercise 3: The character tables for $\mathbf{S}_3$ and $\mathbf{D}_8$ are respectively:

	Γ_1	Γ_2	Γ_3
class	**1**	**(×××)**	**(××)**
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
order	**1**	**4**	**2**

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
class	**1**	**A**	$\mathbf{A}^2$	**B**	**AB**
size	**1**	**2**	**1**	**2**	**2**
χ_1	1	1	1	1	1
χ_2	1	1	1	–1	–1
χ_3	1	–1	1	1	–1
χ_4	1	–1	1	–1	1
χ_5	2	0	–2	0	0
order	**1**	**4**	**2**	**2**	**2**

Let $\Gamma_{ij} = \Gamma_i \times \Gamma_j$.

The character table of $\mathbf{S}_3 \times \mathbf{D}_8$ is as follows. (The table has had to be split vertically.)

class	Γ_{11}	Γ_{21}	Γ_{31}	Γ_{12}	Γ_{22}	Γ_{32}	Γ_{13}	Γ_{23}	Γ_{33}
size	**1**	**2**	**3**	**1**	**2**	**3**	**2**	**4**	**6**
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0	2	–1	0
χ_4	1	1	1	1	1	1	1	1	1
χ_5	1	1	–1	1	1	–1	1	1	–1
χ_6	2	–1	0	2	–1	0	2	–1	0
χ_7	1	1	1	–1	–1	–1	1	1	1
χ_8	1	1	–1	–1	–1	1	1	1	–1
χ_9	2	–1	0	–2	1	0	2	–1	0
χ_{10}	1	1	1	–1	–1	–1	1	1	1
χ_{11}	1	1	–1	–1	–1	1	1	1	–1
χ_{12}	2	–1	0	–2	1	0	2	–1	0
χ_{13}	1	1	1	0	0	0	–2	–2	–2
χ_{14}	1	1	–1	0	0	0	–2	–2	2
χ_{15}	2	–1	0	0	0	0	–4	2	0
order	**1**	**3**	**2**	**4**	**12**	**4**	**2**	**6**	**2**

class	Γ_{14}	Γ_{24}	Γ_{34}	Γ_{15}	Γ_{25}	Γ_{35}
size	2	4	6	2	4	6
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	–1	–1	–1	–1	–1	–1
χ_5	–1	–1	1	–1	–1	1
χ_6	–2	1	0	–2	1	0
χ_7	1	1	1	–1	–1	–1
χ_8	1	1	–1	–1	–1	1
χ_9	2	–1	0	–2	1	0
χ_{10}	–1	–1	–1	1	1	1
χ_{11}	–1	–1	1	1	1	–1
χ_{12}	–2	1	0	2	–1	0
χ_{13}	0	0	0	0	0	0
χ_{14}	0	0	0	0	0	0
χ_{15}	0	0	0	0	0	0
order	2	6	2	2	6	2

Exercise 4:

class	1	A	A^2	B	AB
size	1	2	1	2	2
$\chi_2 + \chi_3 + \chi_4$	3	–1	3	–1	–1
$2\chi_4 + \chi_5$	4	–2	0	–2	2

Exercise 5: The character table for $\mathbf{S}_4$, together with χ_5^2 is:

class	I	(××)(××)	(×××)	(××)	(××××)
size	1	3	8	6	6
χ_1	1	1	1	1	1
χ_2	1	1	1	–1	–1
χ_3	2	2	–1	0	0
χ_4	3	–1	0	–1	1
χ_5	3	–1	0	1	–1
χ_5^2	9	1	0	1	1
order	1	2	3	2	4

$$\langle\chi_5^2|\chi_1\rangle = \frac{9.1.1 + 1.1.3 + 0.1.8 + 1.1.6 + 1.1.6}{24} = 1$$

$$\langle\chi_5^2|\chi_2\rangle = \frac{9.1.1 + 1.1.3 + 0.1.8 + 1.(-1).6 + 1.(-1).6}{24} = 0$$

$$\langle\chi_5^2|\chi_3\rangle = \frac{9.2.1 + 1.2.3 + 0.(-1).8 + 1.0.6 + 1.0.6}{24} = 1$$

$$\langle\chi_5^2|\chi_4\rangle = \frac{9.3.1 + 1.(-1).3 + 0.1.8 + 1.(-1).6 + 1.1.6}{24} = 1$$

$$\langle\chi_5^2|\chi_5\rangle = \frac{9.3.1 + 1.(-1).3 + 0.1.8 + 1.1.6 + 1.(-1).6}{24} = 1$$

Hence $\chi_5^2 = \chi_1 + \chi_3 + \chi_4 + \chi_5$.

Exercise 6: The character table for $\langle B\rangle$ is

	1	B
χ_1	1	1
χ_2	1	–1

The conjugacy classes for $\mathbf{D}_8$ are:

Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
1	A, A^3	A^2	B, A^2B	AB, A^3B

The induced characters are:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
size	**1**	**2**	**1**	**2**	**2**
θ_1	4.1.1	0	0	4.½.1	0
θ_2	4.1.1	0	0	4.½.(–1)	0

That is:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
size	**1**	**2**	**1**	**2**	**2**
θ_1	4	0	0	2	0
θ_2	4	0	0	–2	0

$$\langle\theta_1|\chi_1\rangle = \frac{4.1.1 + 2.1.2}{8} = 1$$

$$\langle\theta_1|\chi_2\rangle = \frac{4.1.1 + 2.(-1).2}{8} = 0 \text{ etc.}$$

So $\langle\theta_i|\chi_j\rangle$ is given in the following table:

	χ_1	χ_2	χ_3	χ_4	χ_5
θ_1	1	0	1	0	1
θ_2	0	1	0	1	1

Hence $\theta_1 = \chi_1 + \chi_3 + \chi_5$ and $\theta_2 = \chi_2 + \chi_4 + \chi_5$.